Problem Set 4 Answers
On Expectation Values

$\langle z\rangle = \int_0^b \Psi^* \, z \, \Psi dz$

$= \int_0^b [2/b]^{1/2} \sin[\pi z/b] z \, [2/b]^{1/2} \sin[\pi z/b] dz$

Physical quantity	z	z^2	p_z	E_{kin}
Operator	z	z^2	$(\hbar/i)\,\partial/\partial z$	$-(\hbar^2/2m)\,\partial^2/\partial z^2$
OpΨ	$z[2/b]^{1/2} \sin[\pi z/b]$	$z^2[2/b]^{1/2} \sin[\pi z/b]$	$(\hbar/i)\,[2/b]^{1/2}\,\pi/b \cos[\pi z/b]$	$(\hbar^2/2m)\,[2/b]^{1/2}\,\pi^2/b^2 \sin[\pi z/b]$
Ψ*OpΨ	$z[2/b] \sin^2[\pi z/b]$	$z^2[2/b] \sin^2[\pi z/b]$	$(\hbar/i)\,[2/b]\,\pi/b \sin[\pi z/b] \cos[\pi z/b]$	$(\hbar^2/2m)\,[2/b]\,\pi^2/b^2 \sin^2[\pi z/b]$
Integration $\int x\sin^2 ax dx = \frac{1}{4}x^2 - (\frac{1}{4}a)x\sin 2ax - (1/8a^2)\cos 2ax$ $\int x^2\sin^2 ax dx = (1/6)x^3 - [(\frac{1}{4}a)x^2 - (1/8a^3)]\sin 2ax - (\frac{1}{4}a^2)x\cos 2ax$ $\int \sin^2(ax)dx = \frac{1}{2}x - (1/4a)\sin(2ax)$	$[2/b]\{ \frac{1}{4}z^2 - (b/4\pi)z\sin[2\pi z/b] - (b^2/8\pi^2)\cos[2\pi z/b]\}$ taken between 0 and b: $[2/b]\{b^2/4 - (b/4\pi)b \sin 2\pi - (b^2/8\pi^2)(\cos 2\pi - \cos 0)\}$ $=[2/b]\{ b^2/4\}$	$[2/b]\{(1/6)z^3 - [(b/4\pi)z^2 + (b/2\pi)^3]\sin[2\pi z/b] + (b/2\pi)^2 z\cos[2\pi z/b]\}$ taken between 0 and b: $[2/b]\{b^3/6 - [b^3/4\pi + (b/2\pi)^3] \sin 2\pi + (b/2\pi)^2[b(\cos 2\pi - 0\cos 0)\}]$ $=[2/b]\{b^3/6 + b^3/4\pi^2 \}$	$(\hbar/i)\,[2\pi/b^2]\frac{1}{2}[b/\pi] \sin^2[\pi z/b]$ taken between 0 and b: $(\hbar/i)[1/b][\sin^2\pi - \sin^2 0]=0$	$(\hbar^2/2m)\,[2\pi^2/b^3]\{z/2 - (\pi/4b)\sin[2\pi z/b]\}$ taken between 0 and b: $(\hbar^2/2m)\,[2\pi^2/b^3]\{b/2 - (\pi/4b)[\sin 2\pi - \sin 0]\}$ $=(\hbar^2/2m)\,[2\pi^2/b^3]\,\{b/2\}$
Expectation value	$\langle z\rangle = b/2$	$\langle z^2\rangle = b^2\{1/3 + (2\pi^2)^{-1}\}$	$\langle p_z\rangle = 0$	$\langle E_{kin}\rangle = (\hbar^2/2m)\pi^2/b^2 = h^2/8mb^2$

The classical average $\langle z^2\rangle$ is $(1/3)b^2$, which is only the first term in the quantum mechanical average.