Answers to Problem Set 6

1. First the function has to be normalized:

$$\Psi = 0.4\psi_1 + 0.5\psi_2 + c_3\psi_3 + 0.3\psi_4$$

by setting $\int \Psi^*\Psi d\tau = 1$

That $\psi_1, \psi_2, \psi_3, \psi_4$ are all the eigenfunctions of an operator **T**, means they are individually normalized and they are also orthogonal to each other since they correspond to <u>different</u> eigenvalues. Integrating leads to

$\int \Psi^*\Psi d\tau = 1 = (0.4)^2 \int \psi_1{}^*\psi_1 d\tau + (0.5)^2 \int \psi_2{}^*\psi_2 d\tau + (c_3)^2 \int \psi_3{}^*\psi_3 d\tau$
$+(0.3)^2 \int \psi_4{}^*\psi_4 d\tau$ + all other integrals $\int \psi_1{}^*\psi_2 d\tau$ etc. are zero.
$c_3{}^*c_3 = 0.5$, $c_3 = 1/\sqrt{2}$

The fraction of the time the eigenvalue a_3 will be observed is $c_3{}^*c_3 = 0.5$ Also a_1, a_2, a_4 will be found the following fractions of time: $(0.4)^2$, $(0.5)^2$, $(0.3)^2$

The expected average of T is given by Postulate 3:

$\int \Psi^*T\Psi d\tau =$

$\int (0.4\psi_1 + 0.5\psi_2 + 1/\sqrt{2}\psi_3 + 0.3\psi_4)^*T (0.4\psi_1 + 0.5\psi_2 + 1/\sqrt{2}\psi_3 + 0.3\psi_4) d\tau$
$= (0.4)^2 a_1 + (0.5)^2 a_2 + 1/2\, a_3 + (0.3)^2 a_4$

The expected average of T^3 is given by Postulate 3:

$\int \Psi^*T^3\Psi d\tau =$

$\int (0.4\psi_1 + 0.5\psi_2 + 1/\sqrt{2}\psi_3 + 0.3\psi_4)^*T^3 (0.4\psi_1 + 0.5\psi_2 + 1/\sqrt{2}\psi_3 + 0.3\psi_4) d\tau$
$= (0.4)^2 (a_1)^3 + (0.5)^2 (a_2)^3 + 1/2\, (a_3)^3 + (0.3)^2 (a_4)^3$
because $T^3\psi_1 = T^2a_1\psi_1 = a_1 T^2\psi_1 = (a_1)^2T\psi_1 = (a_1)^3\psi_1$ and so on.

2. Particle on a line from x= 0 to x= a is described by $\Psi_{t0} = Nx(a-x)$ at t_0

(a) The only possible outcomes of a measurement of energy are the eigenvalues of the Hamiltonian operator for a particle on a line system, which are $n^2h^2/8ma^2$, where n = 1,2,3,4, etc.

Which eigenvalues would be observed will depend on which eigenfunctions are included in $\Psi_{t0} = Nx(a-x)$.

(b) Average of single measurements from multiple copies is given by:

Ave $= \int \Psi_{t0}{}^*\mathcal{H}\Psi_{t0} dx \,/\, \int \Psi_{t0}{}^*\Psi_{t0} dx = \int N^*x(a-x)\mathcal{H}Nx(a-x)dx \,/ \int N^2x^2(a-x)^2dx$

$-(\hbar^2/2m)\, d^2/dx^2\, x(a-x) = ?$ $d/dx[x(a-x)] = -x+(a-x) = a-2x$; $d/dx\,[a-2x] = -2$

$-(\hbar^2/2m)\, d^2/dx^2\, x(a-x) = -(\hbar^2/2m)(-2)$

Ave $= -(\hbar^2/2m) \int -2x(a-x)dx \,/ \int x^2(a-x)^2dx =$ **?**

$\int x(a-x)dx = \left| ax^2/2 - x^3/3 \right|_0^a = a^3/2 - a^3/3 = a^3/6$

$\int x^2(a-x)^2dx = \int a^2x^2 - 2ax^3 + x^4dx = | ax^3/3 - 2ax^4/4 + x^5/5|_0^a = a^4/3 - a^5/2 + a^5/5 = a^5/30$

Ave = $-(\hbar^2/2m)\ -2[\ a^3/6\ ]\ /\ a^5/30 = +5\ (\hbar^2/ma^2)$

Normalization of Ψ_{t0}: $\int N^2x^2(a-x)^2dx = 1 = N^2\ a^5/30$ thus, $N^2 = 30/a^5$
and $N = [30/a^5]^{½}$

(c) Expand in terms of the complete set of eigenfunctions of the system:
$\Psi_{t0} = \sum_n C_n\psi_n$

Operate on both sides with $\int \psi_i * dx$, to get C_i
$\int \psi_n * [30/a^5]^{½} x(a-x)dx = C_n$

since all other integrals in the sum $\sum_i C_i \int \psi_n * \psi_i\, dx$ vanish for $i \neq n$ since the eigenfunctions of the system are orthonornal.

$\int \psi_n * [30/a^5]^{½} x(a-x)dx = C_n = [2/a]^{½} [30/a^5]^{½} \int \sin(n\pi x/a)\, x(a-x)dx$

$C_n = [60/a^6]^{½} \{ \int ax \sin(n\pi x/a)dx - \int x^2 \sin(n\pi x/a)dx \}$

Use known integrals

$$\int x \sin(bx)dx = (1/b^2)\sin(bx) - (x/b)\cos(bx)$$

and $\int x^2 \sin(bx)dx = -[(b^2x^2 - 2)/b^3]\cos(bx) + 2x\sin(bx)/b^2$

After the smoke clears in the integrations, we find that
when n is odd, we get $C_{odd} = 0$
when n is even, we get

$C_n = [60/a^6]^{½}\ 4a^3/(n\pi)^3$

3. Consider a particle of mass M constrained to move on a circle of radius R where its potential energy is zero. The particle is in a physical state that is described by $F(\phi) = A\{\cos 2\phi + 2\cos 3\phi\}$. *Determine the results of the following sets of experiments on this system, that is, determine the typical outcomes of the experiments, the average values of the results:*

(a) The z component of the angular momentum of the system is measured

Derivation of predictions here: $F(\phi)$ is not one of the eigenfunctions of L_z or $\mathcal{H}$.	Observed values here:
Expand $F(\phi)$ in terms of the complete orthonormal set of functions	$2\ \hbar$
$(1/\sqrt{2\pi})\exp[ik\phi]$. $F(\phi) = \sum_k c_k \Psi_k(\phi)$ and find the coefficients:	$-2\ \hbar$
$c_k = \int_0^{2\pi} \Psi_k^*(\phi)\, F(\phi)\, d\phi = \int_0^{2\pi} \Psi_k^*(\phi) A\{\cos 2\phi + 2\cos 3\phi\}\, d\phi$	$3\ \hbar$
Since we can write $\cos 2\phi = ½[\exp(i2\phi) + \exp(-i2\phi)]$ and	$3\ \hbar$
$\cos 3\phi = ½[\exp(i3\phi) + \exp(-i3\phi)]$, then	$3\ \hbar$
$c_k = \int_0^{2\pi} \Psi_k^*(\phi) ½A\{\exp(i2\phi) + \exp(-i2\phi) + 2\exp(i3\phi) + 2\exp(-i3\phi)\}\, d\phi$	$3\ \hbar$
$c_k = (1/\sqrt{2\pi})^{-1} ½A \int_0^{2\pi} \Psi_k^*(\phi)\{\Psi_2(\phi) + \Psi_{-2}(\phi) + 2\Psi_3(\phi) + 2\Psi_{-3}(\phi)\}\, d\phi$	$-3\ \hbar$

$c_k = (1/\sqrt{2\pi})^{-1}½A\ \{\delta_{k,2} + \delta_{k,-2} + 2\delta_{k,3} + 2\delta_{k,-3}\}$ in shorthand where $\delta_{k,2} = 1$ if k=2 or else it is zero, since it was an orthonormal set A can be obtained by integration $\int_0^{2\pi} F^*(\phi)F(\phi)\,d\phi$: $A = [5\pi]^{-½}$ $c_k = (10)^{-½}\{\delta_{k,2} + \delta_{k,-2} + 2\delta_{k,3} + 2\delta_{k,-3}\}$ $c_2^2 = c_{-2}^2 = (1/10)$ $c_3^2 = c_{-3}^2 = (4/10)$	$-3\hbar$ $-3\hbar$ $-3\hbar$ Average = $0\hbar$

(b) The energy of the system is measured

Derivation of predictions here:	Observed values here:
The average value is obtained by $\int_0^{2\pi} F^*(\phi)\ (L_z \text{ or } \mathcal{H})\ F(\phi)\,d\phi$:	$2^2\ (\hbar^2/2MR^2)$
which by above algebra leads to $\sum_k c_k^2 \bullet k\hbar$ or $\sum_k c_k^2 \bullet k^2(\hbar^2/2MR^2)$	$(-2)^2(\hbar^2/2MR^2)$
The observed values should be 10% of the time the eigenvalue for	$3^2\ (\hbar^2/2MR^2)$
k=2, 10% of the time the eigenvalue for k=-2, 40% of the time the	$3^2\ (\hbar^2/2MR^2)$
eigenvalue for k=3, 40% of the time the eigenvalue for k=-3,	$3^2\ (\hbar^2/2MR^2)$
according to the probabilities given by the corresponding c_k^2	$3^2\ (\hbar^2/2MR^2)$
	$(-3)^2(\hbar^2/2MR^2)$
	$(-3)^2(\hbar^2/2MR^2)$
	$(-3)^2(\hbar^2/2MR^2)$
	$(-3)^2(\hbar^2/2MR^2)$
	Average = $8\ (\hbar^2/2MR^2)$

4. A particle of mass *m* in a potential well (with infinitely high walls) in the x dimension is known to be in either the $n = 2$ or $n = 3$ eigenstates with equal probability. The eigenfunctions of these states are $\psi_2(x) = (2/a)^{1/2} \sin[2\pi x/a]$ and $\psi_3(x) = (2/a)^{1/2} \sin[3\pi x/a]$, respectively.

(a) *Write an appropriate wavefunction* Ψ for the system that reflects our knowledge of <u>the state of the system</u>.

According to the conditions of the problem both states are equally probable, thus we need to have the wavefunction be a superposition of $\psi_2(x)$ and $\psi_3(x)$ with coefficients whose absolute squares are equal. $\Psi(x) = c_2\psi_2(x) + c_3\psi_3(x)$ such that $c_2^2 = c_3^2 = 1/2$ since $c_2^2 + c_3^2 = 1$ (normalization). Therefore, $\Psi(x) = (1/\sqrt{2})\{(2/a)^{1/2} \sin[2\pi x/a] + (2/a)^{1/2} \sin[3\pi x/a]\}$ or $\Psi(x) = (1/\sqrt{2})\{(2/a)^{1/2} \sin[2\pi x/a] - (2/a)^{1/2} \sin[3\pi x/a]\}$

(b) *What energies* might be obtained if the energy of the particle is measured?

The energy eigenvalues $2^2h^2/8ma^2$ and $3^2h^2/8ma^2$ only.

(c) *Determine the expected average* of a series of measurements of the energy of the particle.

Postulate 3 says the expected average is $\langle E\rangle = \int_0^a \Psi^*(x)\,\mathcal{H}\,\Psi(x)dx$, since we have already normalized $\Psi(x)$.

$\langle E\rangle = \int_0^a (1/\sqrt{2})\{\psi_2(x) + \psi_3(x)\}^*\,\mathcal{H}(1/\sqrt{2})\{\psi_2(x) + \psi_3(x)\}dx$

$= (1/2)\int_0^a\{\psi_2^*\,\mathcal{H}\,\psi_2dx+\psi_2^*\,\mathcal{H}\,\psi_3dx+\psi_3^*\,\mathcal{H}\,\psi_2dx+\psi_3^*\,\mathcal{H}\,\psi_3dx\}$

$= (1/2)\{E_2 \int_0^a\psi_2^*\psi_2dx + E_3 \int_0^a\psi_2^*\psi_3dx+ E_2 \int_0^a\psi_3^*\psi_2dx+ E_3 \int_0^a\psi_3^*\psi_3dx\}$

$\langle E\rangle = (1/2)\{E_2+ E_3\} = (1/2)\{ 2^2+ 3^2\}h^2/8ma^2$

For the - combination the results are the same as above:

$\langle E\rangle = \int_0^a (1/\sqrt{2})\{\psi_2(x) - \psi_3(x)\}^*\,\mathcal{H}(1/\sqrt{2})\{\psi_2(x) - \psi_3(x)\}dx$

$= (1/2)\{E_2 \int_0^a\psi_2^*\psi_2dx - E_3 \int_0^a\psi_2^*\psi_3dx- E_2 \int_0^a\psi_3^*\psi_2dx+ E_3 \int_0^a\psi_3^*\psi_3dx\}$

$\langle E\rangle = (1/2)\{E_2+ E_3\} = (1/2)\{ 2^2+ 3^2\}h^2/8ma^2$

(d) *Write the equation* that shows how the <u>expected</u> <u>mean square deviation</u> of any series of measurements of the energy of the particle can be calculated.

The operator for the square of the deviation in measurements of energy is

$Op = (\mathcal{H} - \langle E\rangle)^2$

Postulate 3 gives the average, thus the mean square deviation

$= \int_0^a (1/\sqrt{2})\{\psi_2(x) + \psi_3(x)\}^* (\mathcal{H} - \langle E\rangle)^2 (1/\sqrt{2})\{\psi_2(x) + \psi_3(x)\}dx$

(e) *Carry out the solution* of (e), and then from the final result, determine the expected standard deviation of the series of measurements.

mean square dev $=\int_0^a (1/\sqrt{2})\{\psi_2(x) + \psi_3(x)\}^* (\mathcal{H} - \langle E\rangle)^2 (1/\sqrt{2})\{\psi_2(x) + \psi_3(x)\}dx$

$=\int_0^a (1/\sqrt{2})\{\psi_2(x) + \psi_3(x)\}^* (\mathcal{H}^2 - 2\langle E\rangle\,\mathcal{H}+\langle E\rangle^2) (1/\sqrt{2})\{\psi_2(x) + \psi_3(x)\}dx$

Note that $\mathcal{H}^2\psi_2(x) = \mathcal{H}E_2\psi_2(x) = E_2\,\mathcal{H}\psi_2(x) = E_2^2\psi_2(x)$

and $\langle E\rangle\,\mathcal{H}\psi_2(x) = \langle E\rangle E_2\psi_2(x)$ and $\langle E\rangle^2\psi_2(x) = \langle E\rangle^2\psi_2(x)$ since $\langle E\rangle^2$ is a number.

mean square dev =

$(1/2)\int_0^a\{\psi_2 + \psi_3\}^* (E_2^2\psi_2+E_3^2\psi_3 - 2\langle E\rangle E_2\psi_2 - 2\langle E\rangle E_3\psi_3 + \langle E\rangle^2\psi_2 + \langle E\rangle^2\psi_3\}dx$

$= (1/2)\{ E_2^2+ E_3^2- 2\langle E\rangle(E_2+ E_3)+2\langle E\rangle^2\} = (1/2)\{ E_2^2+ E_3^2- 4\langle E\rangle^2+2\langle E\rangle^2\}$

$= (1/2)\{ E_2^2+ E_3^2\} - \langle E\rangle^2 = [(1/2)\{2^4+3^4\} - \{(2^2+3^2)/2\}^2]\{h^2/8ma^2\}^2$

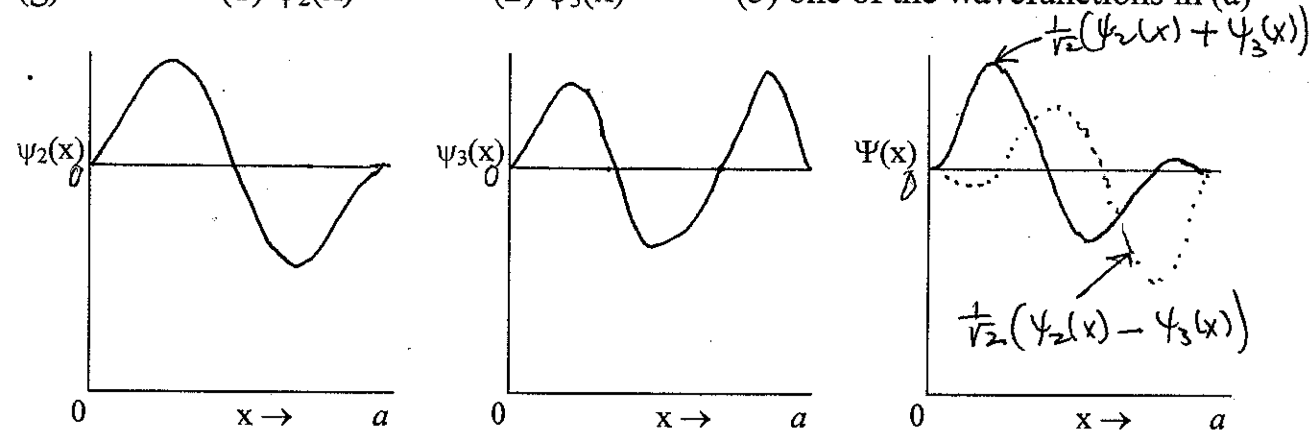
$= [25/4] \{h^2/8ma^2\}^2$

standard deviation is the square root of this $=(5/2)(h^2/8ma^2)$

(e) Illustrate a typical table of results from 10 such measurements. *Fill in* the column "Results" . What is the probability of each outcome?

	Result	Deviation		Probability
1	$2^2h^2/8ma^2$	$-2,5h^2/8ma^2$		1/2
2	$2^2h^2/8ma^2$			
3	$3^2h^2/8ma^2$	$+2,5h^2/8ma^2$		1/2
4	$2^2h^2/8ma^2$			
5	$3^2h^2/8ma^2$			
6	$3^2h^2/8ma^2$			
7	$3^2h^2/8ma^2$			
8	$2^2h^2/8ma^2$			
9	$3^2h^2/8ma^2$			
10	$2^2h^2/8ma^2$			
Ave	$6.5\,h^2/8ma^2$			

(g) *Sketch* (1) $\psi_2(x)$ (2) $\psi_3(x)$ (3) one of the wavefunctions in (a)



(h) Suppose an electron is contained in a two-dimensional potential well (with infinitely high walls) whose shape is that of a rectangular sheet with dimensions $a{\times}b$. *Write the Schrodinger equation* that needs to be solved for this system.

$$\mathcal{H}\Psi(x,y) = -\frac{\hbar^2}{2m_e}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\Psi(x,y) = E\,\Psi(x,y)$$

(i) *Show* that the method of separation of variables may be used to solve this problem, i.e., to find the eigenfunctions and eigenvalues.

Let $\Psi(x, y) = F(x) \cdot G(y)$

$$\frac{-\hbar^2}{2m_e}\left[\frac{d^2}{dx^2} F(x)\cdot G(y) + \frac{d^2}{dy^2} F(x)\cdot G(y)\right] = E\, F(x)\cdot G(y)$$

Dividing both sides by $F(x)\cdot G(y)$ after equating

$$\frac{-\frac{\hbar^2}{2m_e}\left[G(y)\cdot \frac{d^2F(x)}{dx^2} + F(x)\frac{d^2G(y)}{dy^2}\right]}{F(x)\cdot G(y)} = \frac{E\, F(x)\cdot G(y)}{F(x)\cdot G(y)} = E$$

$$\frac{-\hbar^2}{2m_e}\left[\underbrace{\frac{d^2F(x)}{dx^2}\cdot\frac{1}{F(x)}} + \underbrace{\frac{1}{G(y)}\frac{d^2G(y)}{dy^2}}\right] = E$$

Since each of these ~~can~~ independently take any x values unrelated to y values, ~~and~~ yet the sum has always to be the same constant value E then each term in the sum must itself be a constant, independent of the value of x or y.

$$\underbrace{\frac{-\hbar^2}{2m_e}\frac{d^2F(x)}{F(x)dx^2}}_{\text{can solve this}} = A \qquad \underbrace{\frac{-\hbar^2}{2m_e}\frac{1}{G(y)}\frac{d^2G(y)}{dy}}_{\text{can solve this}} = B, \text{ such that } A+B=E$$

(j) Given the results of your proof above, *write down* the possible energy *eigenfunctions* for an electron confined to a sheet with dimensions $a \times b$. Given the results of your proof above, *write down* the corresponding *energy eigenvalues* opposite the eigenfunction

Solving $\frac{-\hbar^2}{2m_e}\frac{d^2F(x)}{F(x)\,dx^2} = A$ or $\frac{-\hbar^2}{2m_e}\frac{d^2F(x)}{dx^2} = A\,F(x)$

eigenvalues are already known, as seen in part (b) of this exam: $A = n_x^2 h^2/8ma^2$ $\quad B = \frac{n_y^2 h^2}{8mb^2}$

Both equations have the same form, therefore the solutions are the same form.

$$E = A + B = \frac{n_x^2 h^2}{8m_e a^2} + \frac{n_y^2 h^2}{8m_e b^2} \quad \text{where } n_x = 1, 2, 3, \ldots \quad n_y = 1, 2, 3, \ldots$$

Functions are therefore $F(x)\cdot G(y)$ or

$$\Psi(x, y) = \sqrt{\frac{2}{a}}\sin\left(\frac{n_x\pi}{a}x\right)\cdot\sqrt{\frac{2}{b}}\sin\left(\frac{n_y\pi}{b}y\right)$$