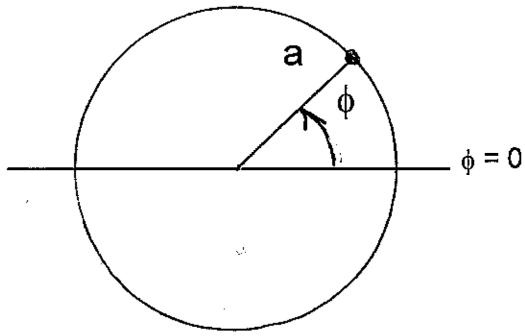
2. Given the physical system of a particle of mass M on a circle of radius a, that is, the potential is infinite everywhere except on the circle itself, where V = a constant, call it zero. The operator for the z component of the angular momentum of the particle is $(\hbar/i)\,(\partial/\partial\phi)$ and the kinetic energy operator is derived from $-(\hbar^2/2M)\,\{(\partial^2/\partial x^2) + (\partial^2/\partial x^2)\}$ $= -(\hbar^2/2Ma^2)\,(\partial^2/\partial\phi^2)$ for the case where $x^2 + y^2 = a^2$



(a) Write the equation that has to be satisfied by the eigenfunctions [call them $F(\phi)$] of the z component of the angular momentum of the particle.

(b) Find the eigenfunctions of the z component of the angular momentum of the particle. Justify whatever you do, using the postulates of quantum mechanics, or mathematical identities

(c) Suppose this system is in the state described by the function $\Psi(\phi) = \pi^{-\frac{1}{2}} [2\cos^2\phi - 1]$. Test and verify that this is an <u>acceptable</u> state function (including normalization) for this physical system.

(d) Suppose the z component of the angular momentum is measured for the system while it is in the state given by the function $\Psi(\phi) = \pi^{-\frac{1}{2}} [2\cos^2\phi - 1]$. What is the expected average of the measured values?

(e) Using mathematical identities (see last page), and the idea of superposition of states, find out which ones of the known eigenfunctions of the angular momentum of the particle on a circle are included in the state function $\Psi(\phi) = \pi^{-\frac{1}{2}} [2\cos^2\phi - 1]$. Be careful doing this, your answers to (f) and (g) will depend on it.

(f) While the system is in the state described by $\Psi(\phi) = \pi^{-1/2} [2\cos^2\phi - 1]$, give the possible values that could be the result of measurements of its z component of angular momentum. Of one hundred such measurements, how many times would one expect to find your first values? your second, ... etc.

value	number of times found, out of 100 measurements

(g) Determine whether the function $\Psi(\phi) = \pi^{-1/2} [2\cos^2\phi - 1]$ is an eigenfunction of energy.

(h) While the system is in the state described by $\Psi(\phi) = \pi^{-1/2} [2\cos^2\phi - 1]$, give the possible values that could be the result of measurements of its energy.

value	number of times found, out of 100 measurements

List of possibly useful integrals that will be provided with each exam

$\int \sin(ax)dx = -(1/a)\cos(ax)$
$\int \cos(ax)dx = (1/a)\sin(ax)$
$\int \sin^2(ax)dx = ½\,x - (1/4a)\sin(2ax)$
$\int \sin^4(ax)dx = 3x/8 - (1/4a)\sin(2ax) + (1/32a)\sin(4ax)$
$\int \cos^2(ax)dx = ½\,x + (1/4a)\sin(2ax)$
$\int \cos^4(ax)dx = 3x/8 + (1/4a)\sin(2ax) + (1/32a)\sin(4ax)$
$\int \sin(ax)\sin(bx)dx = [1/2(a-b)]\sin[(a-b)x)] - [1/2(a+b)]\sin[(a+b)x)], \quad a^2 \neq b^2$
$\int \cos(ax)\cos(bx)dx = [1/2(a-b)]\sin[(a-b)x)] + [1/2(a+b)]\sin[(a+b)x)], \quad a^2 \neq b^2$
$\int x\sin(ax)dx = (1/a^2)\sin(ax) - (x/a)\cos(ax)$
$\int x\cos(ax)dx = (1/a^2)\cos(ax) + (x/a)\sin(ax)$
$\int x^2\cos(ax)dx = [(a^2x^2 - 2)/a^3]\sin(ax) + 2x\cos(ax)/a^2$
$\int x^2\sin(ax)dx = -[(a^2x^2 - 2)/a^3]\cos(ax) + 2x\sin(ax)/a^2$
$\int x\sin^2(ax)dx = x^2/4 - x\sin(2ax)/4a - \cos(2ax)/8a^2$
$\int x^2\sin^2(ax)dx = x^3/6 - [x^2/4a - 1/8a^3]\sin(2ax) - x\cos(2ax)/4a^2$
$\int x\cos^2(ax)dx = x^2/4 + x\sin(2ax)/4a + \cos(2ax)/8a^2$
$\int x^2\cos^2(ax)dx = x^3/6 + [x^2/4a - 1/8a^3]\sin(2ax) + x\cos(2ax)/4a^2$
$\int x\exp(ax)dx = \exp(ax)\,(ax-1)/a^2$
$\int x\exp(-ax)dx = \exp(-ax)\,(-ax-1)/a^2$
$\int x^2\exp(ax)dx = \exp(ax)\,[x^2/a - 2x/a^2 + 2/a^3]$
$\int x^m\exp(ax)dx = \exp(ax)\sum_{r=0 \text{ to } m}(-1)^r\, m!x^{m-r}/(m-r)!a^{r+1}$

$\int_0^\infty x^n\exp(-ax)dx = n!/a^{n+1} \qquad a > 0$, n positive integer
$\int_0^\infty x^2\exp(-ax^2)dx = (1/4a)(\pi/a)^{½} \qquad a > 0$
$\int_0^\infty x^{2n}\exp(-ax^2)dx = (1\cdot3\cdot5\cdot\ldots\cdot(2n-1)/(2^{n+1}a^n)(\pi/a)^{½} \qquad a > 0$
$\int_0^\infty x^{2n+1}\exp(-ax^2)dx = n!/2a^{n+1} \qquad a > 0$, n positive integer
$\int_0^\infty \exp(-a^2x^2)dx = (1/2a)\,(\pi)^{½} \qquad a > 0$
$\int_0^\infty \exp(-ax)\cos(bx)dx = a/(a^2+b^2) \qquad a > 0$
$\int_0^\infty \exp(-ax)\sin(bx)dx = b/(a^2+b^2) \qquad a > 0$
$\int_0^\infty x\exp(-ax)\sin(bx)dx = 2ab/(a^2+b^2)^2 \qquad a > 0$
$\int_0^\infty x\exp(-ax)\cos(bx)dx = (a^2-b^2)/(a^2+b^2)^2 \qquad a > 0$
$\int_0^\infty \exp(-a^2x^2)\cos(bx)dx = [(\pi)^{½}/2a]\cdot\exp[-b^2/4a^2] \qquad ab \neq 0$

Some useful identities:
$\sin(2x) = 2\sin x\cos x \qquad \cos(2x) = \cos^2x - \sin^2x = 2\cos^2x - 1$
$e^{ix} = \cos x + i\sin x$; $\quad e^{-ix} = \cos x - i\sin x$; $\quad$ from which, $\cos(x) = ½\{\exp[i\,x] + \exp[-i\,x]\}$