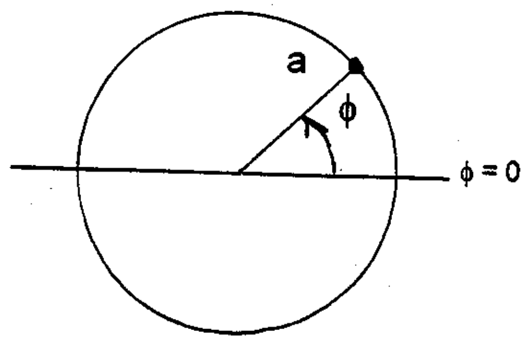
2. Given the physical system of a particle of mass M on a circle of radius a, that is, the potential is infinite everywhere except on the circle itself, where V = a constant, call it zero. The operator for the z component of the angular momentum of the particle is $(\hbar/i)(\partial/\partial\phi)$ and the kinetic energy operator is derived from $-(\hbar^2/2M)\{(\partial^2/\partial x^2) + (\partial^2/\partial x^2)\}$
$= -(\hbar^2/2Ma^2)(\partial^2/\partial\phi^2)$ for the case where $x^2 + y^2 = a^2$



(a) Write the equation that has to be satisfied by the eigenfunctions [call them $F(\phi)$] of the z component of the angular momentum of the particle.

From the postulate on eigenfunctions and eigenvalues, using the operator $(\hbar/i)(\partial/\partial\phi)$
the equation is: $(\hbar/i)(\partial/\partial\phi) F(\phi) = q F(\phi)$ where q is a number, the eigenvalue

(b) Find the eigenfunctions of the z component of the angular momentum of the particle. Justify whatever you do, using the postulates of quantum mechanics, or mathematical identities

$(\hbar/i)(\partial/\partial\phi) F(\phi) = q F(\phi)$ we see that the function $F(\phi)$ has to be one that reverts to itself upon taking the first derivative. $F(\phi)$ must be an exponential function with an exponent that is proportional to ϕ. Let $F(\phi) = A \exp[i b\phi]$, where A and b are constants.
$(\hbar/i)(\partial/\partial\phi) F(\phi) = (\hbar/i)(\partial/\partial\phi) A \exp[i b\phi] = (\hbar/i)(i b) A \exp[i b\phi] = \hbar b A \exp[i b\phi] = \hbar b F(\phi)$
$(\hbar/i)(\partial/\partial\phi) F(\phi) = \hbar b F(\phi)$ means that the eigenvalues of the z component of the angular momentum are $\hbar b$. $F(\phi)$ must be single-valued, continuous and finite. To be single-valued, $F(\phi) = F(2\pi+\phi) = F(4\pi+\phi) = F(6\pi+\phi) = F(8\pi+\phi) = \ldots$ since the same point in space is given by all these different angles. That is, $\exp[i b\phi] = \exp[i b(2\pi+\phi)] = \exp[i b(4\pi+\phi)] = \ldots$
or $\exp[i b\phi] = \exp[i b(2\pi+\phi)] = \exp[i b\phi] \bullet \exp[i b2\pi] = \exp[i b\phi] \bullet \exp[i b4\pi] = \ldots$
Therefore, the equality holds for arbitrary values of ϕ if and only if
$\exp[i b2\pi] = \exp[i b4\pi] = \ldots = 1$, or $\cos[b2\pi] + i\sin[b2\pi] = 1 = \cos[b4\pi] + i\sin[b4\pi] = \ldots$
Of course, these hold only for b = positive or negative integer.
Thus, the eigenvalue $b\hbar$ can be $0\hbar$, $1\hbar$, $-1\hbar$, $2\hbar$, $-2\hbar$, ... etc.
and the eigenfunctions can be $\{\exp[0], \exp[i\phi], \exp[-i\phi], \exp[i2\phi], \exp[-i2\phi], \} \bullet 1/\sqrt{2\pi}$.

(c) Suppose this system is in the state described by the function $\Psi(\phi) = \pi^{-½} [2\cos^2\phi -1]$. Test and verify that this is an <u>acceptable</u> state function (including normalization) for this physical system.

$\Psi(\phi) = \pi^{-½} [2\cos^2\phi -1]$ is *finite* since $\cos^2\phi$ can only be between 0 and 1.
It is *continuous* since the $\cos^2\phi$ function is itself continuous.
The *derivative* $d\Psi/d\phi$ *is continuous* too: $d\Psi/d\phi = -\pi^{-½}2\sin2\phi$ and sine function is continuous.
$\pi^{-½} [2\cos^2\phi -1]$ is *single-valued* since $\cos^2(\phi+2\pi) = \cos^2(\phi+4\pi) = \cos^2\phi$, etc. single value for same point in space.
We can check whether it is *normalized* by integrating over the entire range:
$\int_0^{2\pi} \Psi(\phi)^* \Psi(\phi)d\phi = 1?$ $\quad \int_0^{2\pi} \pi^{-1} [2\cos^2\phi -1]^2 d\phi = \int_0^{2\pi} \pi^{-1} [4\cos^4\phi -4\cos^2\phi +1] d\phi$
We look up the integral of $\cos^4\phi\, d\phi$ and note that at the two limits $\phi=0$ and $\phi=2\pi$, the sin(0) =0 and sin(integer•2π) = 0 so the sin terms resulting after integration all go to zero at the limits.
We are left with $\pi^{-1}[4(3\phi/8) -4(\phi/2) +\phi]$ to be evaluated at $\phi=0$ and $\phi=2\pi$. Thus, we find
$\int_0^{2\pi} \Psi(\phi)^* \Psi(\phi)d\phi = \pi^{-1}[4(6\pi/8) -4(2\pi/2) +2\pi] = 1.$ The function is indeed normalized.

(d) Suppose the z component of the angular momentum is measured for the system while it is in the state given by the function $\Psi(\phi) = \pi^{-½} [2\cos^2\phi -1]$. What is the expected average of the measured values?

The postulate on expectation values says
expected average $= \int_0^{2\pi} \Psi(\phi)^* \mathrm{Op}\Psi(\phi)d\phi = \int_0^{2\pi} \Psi(\phi)^* (\hbar/i) (\partial /\partial\phi)\Psi(\phi)d\phi$
$= \int_0^{2\pi} \pi^{-½} [2\cos^2\phi -1] (\hbar/i) (\partial /\partial\phi)\pi^{-½} [2\cos^2\phi -1] d\phi = \pi^{-1}(\hbar/i)\int_0^{2\pi}[2\cos^2\phi -1] 4\cos\phi(-\sin\phi) d\phi$
$= \pi^{-1}(\hbar/i)\int_0^{2\pi} \{-8\cos^3\phi \sin\phi+4\cos\phi \sin\phi) d\phi$. This is of the form $\int x^3dx$ and $\int xdx$ taken between x= cos(0)=1 and x=cos(2π)=1. Since the upper and lower limits of x are the same, the integral is ZERO!
expected average = zero

(e) Using mathematical identities(see last page), and the idea of superposition of states, find out which ones of the known eigenfunctions of the angular momentum of the particle on a circle are included in the state function $\Psi(\phi) = \pi^{-½} [2\cos^2\phi -1]$. Be careful doing this, your answers to (f) and (g) will depend on it.

$\Psi(\phi) = \pi^{-½} [2\cos^2\phi -1]$ can be written as a linear combination of the eigenfunctions $F(\phi)$ to find out which eigenstates are combined within it:
$\Psi(\phi) = \pi^{-½} [2\cos^2\phi -1] = \{ c_0 1+ c_1 \exp[i \phi] + c_{-1} \exp[-i \phi]+ c_2\exp[i2 \phi]+ c_{-2}\exp[-i2 \phi]+.. \} \bullet 1/\sqrt{2\pi}$
Before we do that, follow the suggestion of using the mathematical identities:
Using the identity $\cos(2\phi) =[2\cos^2\phi -1]$ and the identity $\cos(2\phi) = ½ \{ \exp[i2 \phi]+ \exp[-i2 \phi] \}$
we find $\Psi(\phi) = \pi^{-½} [2\cos^2\phi -1] = ½\pi^{-½} \{ \exp[i2 \phi]+ \exp[-i2 \phi] \}$. So we can merely look at it and immediately identify the previously unknown coefficients
$c_{all} = 0$, except for $c_2 = 1/\sqrt{2}$ and $c_{-2} = 1/\sqrt{2}$. That is, only the two eigenfunctions corresponding to eigenvalues $2\hbar$ and $-2\hbar$ are contained in the state described by $\Psi(\phi) = \pi^{-½} [2\cos^2\phi -1]$ in equal parts!

(f) While the system is in the state described by $\Psi(\phi) = \pi^{-1/2}\,[2\cos^2\phi - 1]$, give the possible values that could be the result of measurements of its z component of angular momentum. Of one hundred such measurements, how many times would one expect to find your first values? your second, ... etc.

value	number of times found, out of 100 measurements
$2\hbar$	50
$-2\hbar$	50
all others	0

(g) Determine whether the function $\Psi(\phi) = \pi^{-1/2}\,[2\cos^2\phi - 1]$ is an eigenfunction of energy.

To find out if it is an eigenfunction of energy, we test whether it satisfies the eigenvalue eigenfunction equation for energy:

$$-(\hbar^2/2Ma^2)\,(\partial^2/\partial\phi^2)\,\Psi(\phi) = E\,\Psi(\phi)$$ Does it? Let us see.

We already found this function can also be written as $\Psi(\phi) = \pi^{-1/2}\cos(2\phi)$

$(\partial^2/\partial\phi^2)\cos(2\phi) = (\partial/\partial\phi)[-2\sin(2\phi)] = -2[2\cos(2\phi)] = -4\cos(2\phi)$ so we do get the same funciton back again! The answer is yes, it is an eigenfunction of energy. Now we can find the eigenvalue corresponding to this eigenfunction of energy:

$-(\hbar^2/2Ma^2)\,(\pi^{-1/2})(-4)\cos(2\phi) = E\,\pi^{-1/2}\cos(2\phi)$

We see that $E = 4(\hbar^2/2Ma^2)$.

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Alternatively, we could have used what we found in part (e):

$\Psi(\phi) = \pi^{-1/2}\,[2\cos^2\phi - 1] = \tfrac{1}{2}\pi^{-1/2}\,\{\exp[i2\phi] + \exp[-i2\phi]\}$

which shows a linear combination of the previously known energy eigenfunctions of the particle on a circle that have the common eigenvalue $2^2\,(\hbar^2/2Ma^2)$. Any linear combination of degenerate eigenfucntions of an operator also satisfies the equation because the common eigenvalue can be factored out. So $\Psi(\phi) = \pi^{-1/2}\,[2\cos^2\phi - 1]$ is an eigenfunction of energy with eigenvalue $2^2\,(\hbar^2/2Ma^2)$.

(h) While the system is in the state described by $\Psi(\phi) = \pi^{-1/2}\,[2\cos^2\phi - 1]$, give the possible values that could be the result of measurements of its energy.

value	number of times found, out of 100 measurements
$2^2\,(\hbar^2/2Ma^2)$	100
all others	0