1. INTRODUCTION TO QUANTUM MECHANICS

2. ANGULAR MOMENTUM

2.1 Classical Mechanics → Quantum Mechanics

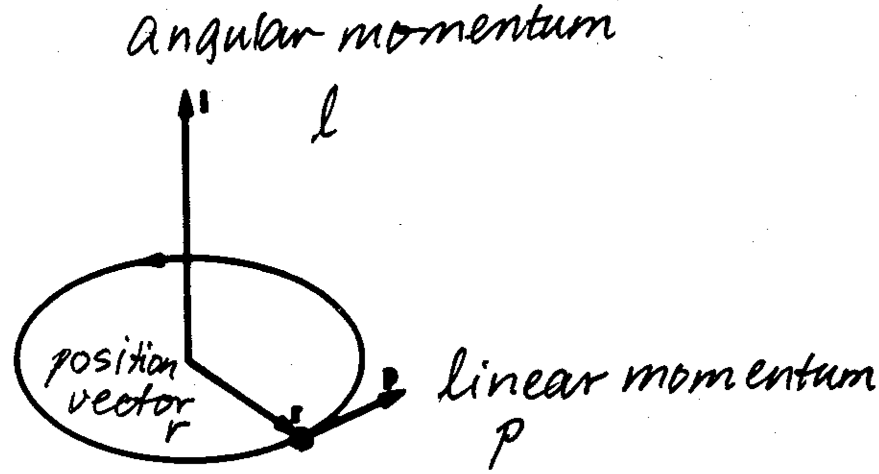
2.2 Commutation Rules of Angular Momentum

2.3 Example: Particle on a Sphere

2.4 Example: The Rigid Rotor

2.5 Eigenfunctions of Angular Momentum

ANGULAR MOMENTUM



The classical definition of angular momentum, $\mathbf{l} = \mathbf{r} \wedge \mathbf{p}$.

CLASSICAL MECHANICS

$$\boldsymbol{\ell} = \mathbf{r} \times p \quad \text{or} \quad \mathbf{r} \wedge p$$

"cross product" or "vector product"

position vector

$$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Linear momentum

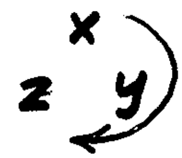
$$\mathbf{p} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$$

angular momentum

$$\ell = (yP_z - zP_y)\hat{i} + (zP_x - xP_z)\hat{j} + (xP_y - yP_x)\hat{k}$$

$$r \times p = \begin{vmatrix} \hat{i} & \hat{j} & k \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix}$$

$$l_x = yp_z - zp_y$$



$$l_y = zp_x - xp_z$$

$$l_z = xp_y - yp_x$$

$$\boldsymbol{l} \cdot \boldsymbol{l} = l^2 = l_x^2 + l_y^2 + l_z^2$$

dot product
or scalar product

QUANTUM MECHANICS

Replace p_x, p_y, and p_z by operators to find

$$l_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$l_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$l_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Commutation Rules:

$$[l_y, l_z] = i\hbar l_x$$

$$[l_z, l_x] = i\hbar l_y$$

$$[l_x, l_y] = i\hbar l_z$$

1. INTRODUCTION TO QUANTUM MECHANICS

2. ANGULAR MOMENTUM

2.1 Classical Mechanics → Quantum Mechanics

2.2 Commutation Rules of Angular Momentum

2.3 Example: Particle on a Sphere

2.4 Example: The Rigid Rotor

2.5 Eigenfunctions of Angular Momentum

$$l_x = y p_z - z p_y$$

x
z y

$$l_y = z p_x - x p_z$$

$$l_z = x p_y - y p_x$$

$$\mathbf{l} \cdot \mathbf{l} = l^2 = l_x^{\,2} + l_y^{\,2} + l_z^{\,2}$$

dot product
or scalar product

QUANTUM MECHANICS

Replace p_x, p_y, and p_z by operators to find

$$l_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$l_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$l_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Commutation Rules:

$$[l_y, l_z] = i\hbar l_x$$

$$[l_z, l_x] = i\hbar l_y$$

x
z y

$$[l_x, l_y] = i\hbar l_z$$

How did we get the commutators?

$$[l_x, l_y] = \frac{\hbar}{i}\cdot\frac{\hbar}{i}\left\{ \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right\}$$

$$= \frac{\hbar}{i}\frac{\hbar}{i}\left\{ y\frac{\partial}{\partial z}\, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\, z\frac{\partial}{\partial y} + z\frac{\partial}{\partial y}\, x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\, y\frac{\partial}{\partial z}\right\}$$ all others drop out

$$y\frac{\partial}{\partial z}\, z\frac{\partial}{\partial x}\psi(x,y,z) = y\frac{\partial}{\partial z}\left(z\underbrace{\frac{\partial\psi}{\partial x}}_{\text{itself is a function of } z}\right) = y\frac{\partial\psi}{\partial x} + \cancel{yz\frac{\partial^2\psi}{\partial z\,\partial x}}$$

$$z\frac{\partial}{\partial y}\, x\frac{\partial\psi}{\partial z} = \cancel{zx\frac{\partial^2\psi}{\partial y\,\partial z}}$$

On the other hand, the negative terms:

$$x\frac{\partial}{\partial z}\, z\frac{\partial\psi(x,y,z)}{\partial y} = x\frac{\partial}{\partial z}\left(z\underbrace{\frac{\partial\psi}{\partial y}}_{\text{a function of } z}\right) = x\frac{\partial\psi}{\partial y} + \cancel{xz\frac{\partial^2\psi}{\partial z\,\partial y}}$$

$$z\frac{\partial}{\partial x}\, y\frac{\partial\psi}{\partial z} = \cancel{zy\frac{\partial^2\psi}{\partial x\,\partial z}}$$

Leaving only:

$$[l_x, l_y] = \frac{\hbar}{i}\underbrace{\frac{\hbar}{i}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)}_{-l_z} = i\hbar\, l_z$$

The others can be shown similarly.

$$[l^2, l_x] = l^2 l_x - l_x l^2 = (l_x^2 + l_y^2 + l_z^2)\, l_x - l_x (l_x^2 + l_y^2 + l_z^2)$$

$$= \cancel{l_x^3} + l_y \underbrace{l_y l_x}_{\substack{l_x l_y \\ -i\hbar l_z}} + l_z \underbrace{l_z l_x}_{\substack{l_x l_z \\ +i\hbar l_y}} - \cancel{l_x^3} - \underbrace{l_x l_y}_{\substack{l_y l_x \\ +i\hbar l_z}} l_y - \underbrace{l_x l_z}_{\substack{l_z l_x \\ -i\hbar l_y}} l_z$$

$$= \cancel{l_y l_x l_y} - \cancel{i\hbar l_y l_z} + \cancel{l_z l_x l_z} + \cancel{i\hbar l_z l_y} - \cancel{l_y l_x l_y} - \cancel{i\hbar l_z l_y} - \cancel{l_z l_x l_z} + \cancel{i\hbar l_y l_z}$$

$$[l^2, l_x] = 0$$

Similarly, one can show that

$$[l^2, l_y] = 0$$

$$[l^2, l_z] = 0$$

This means that l^2 and any one of its components are simultaneously knowable, or

$$\sigma_{l^2} \cdot \sigma_{l_x} \geq 0 \qquad \sigma_{l^2} \cdot \sigma_{l_y} \geq 0$$

$$\text{and} \quad \sigma_{l^2} \cdot \sigma_{l_z} \geq 0$$

But, since $[l_x, l_y] = i\hbar l_z \neq 0$, then

$$\sigma_{l_x} \cdot \sigma_{l_y} \geq \frac{1}{2} \left\langle \frac{[l_x, l_y]}{i} \right\rangle$$

or $\quad \sigma_{l_x} \cdot \sigma_{l_y} \geq \frac{\hbar}{2} \langle l_z \rangle$

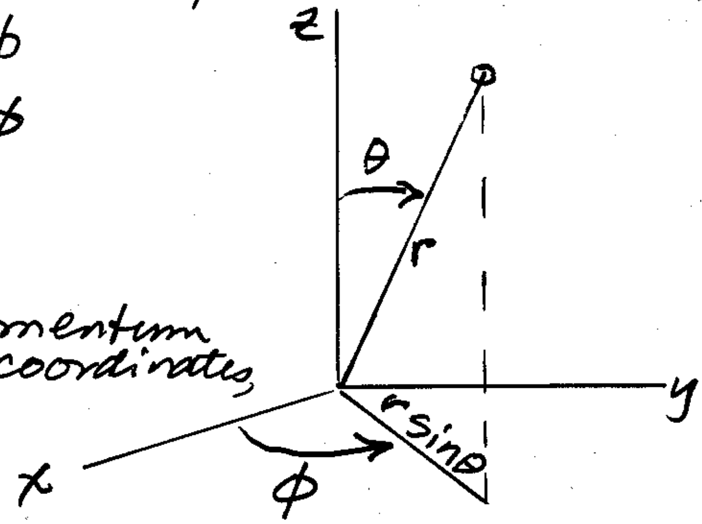
We know the form of these operators in cartesian coordinates x, y, z.

If we use the spherical polar coordinates:

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$



From the angular momentum operators in x, y, z, coordinates, With some effort, one can derive:

$$\ell^2 = -\hbar^2\left\{\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right\}$$

$$\ell_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\ell_y = \frac{\hbar}{i}\left\{\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right\}$$

$$\ell_x = -\frac{\hbar}{i}\left\{\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right\}$$

The angular momentum operators in r, θ, ϕ, coordinates

1. INTRODUCTION TO QUANTUM MECHANICS

2. ANGULAR MOMENTUM

2.1 Classical Mechanics → Quantum Mechanics

2.2 Commutation Rules of Angular Momentum

2.3 Example: Particle on a Sphere

2.4 Example: The Rigid Rotor

2.5 Eigenfunctions of Angular Momentum

EXAMPLE: Particle on a sphere

$V = \infty$ everywhere except on the surface of the sphere for one particle of mass M

$$H = \frac{-\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \underset{\substack{(\text{zero on sphere} \\ \infty \text{ elsewhere})}}{V}$$

Use spherical polar coordinates

r = radius of sphere = constant

$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

$$H = \frac{-\hbar^2}{2M}\left\{\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}$$

for r = constant

$$H = \frac{-\hbar^2}{2Mr^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}$$

SEPARATION OF VARIABLES may be possible:

Let the solutions to the equation:

$$H\,Y(\theta,\phi) = E\,Y(\theta,\phi)$$

be a SIMPLE PRODUCT of functions:

$$Y(\theta,\phi) = \Theta(\theta)\cdot\Phi(\phi)$$

Try it and see if it satisfies the Schrödinger equation:

$$\frac{-\hbar^2}{2Mr^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\Theta(\theta)\,\Phi(\phi) = E\,\Theta(\theta)\,\Phi(\phi)$$

Multiply both sides by $\dfrac{\sin^2\theta\, 2Mr^2}{-\hbar^2\,\Theta(\theta)\,\Phi(\phi)}$

$$\frac{\left(\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\phi^2}\right)\Theta(\theta)\,\Phi(\phi)}{\Theta(\theta)\,\Phi(\phi)} = E\sin^2\theta\,\frac{2Mr^2}{-\hbar^2}$$

$$\underbrace{\frac{\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta)}{\Theta(\theta)}+\frac{2Mr^2E\sin^2\theta}{\hbar^2}}_{\text{Therefore this must be } +m^2} + \underbrace{\frac{\frac{\partial^2\Phi(\phi)}{\partial\phi^2}}{\Phi(\phi)}}_{} = 0$$

From particle on a ring we already know this to be $-m^2$ where $m = 0, \pm 1, \pm 2 \ldots$

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

multiply by $\dfrac{\Theta(\theta)}{\sin^2\theta}$ to get:

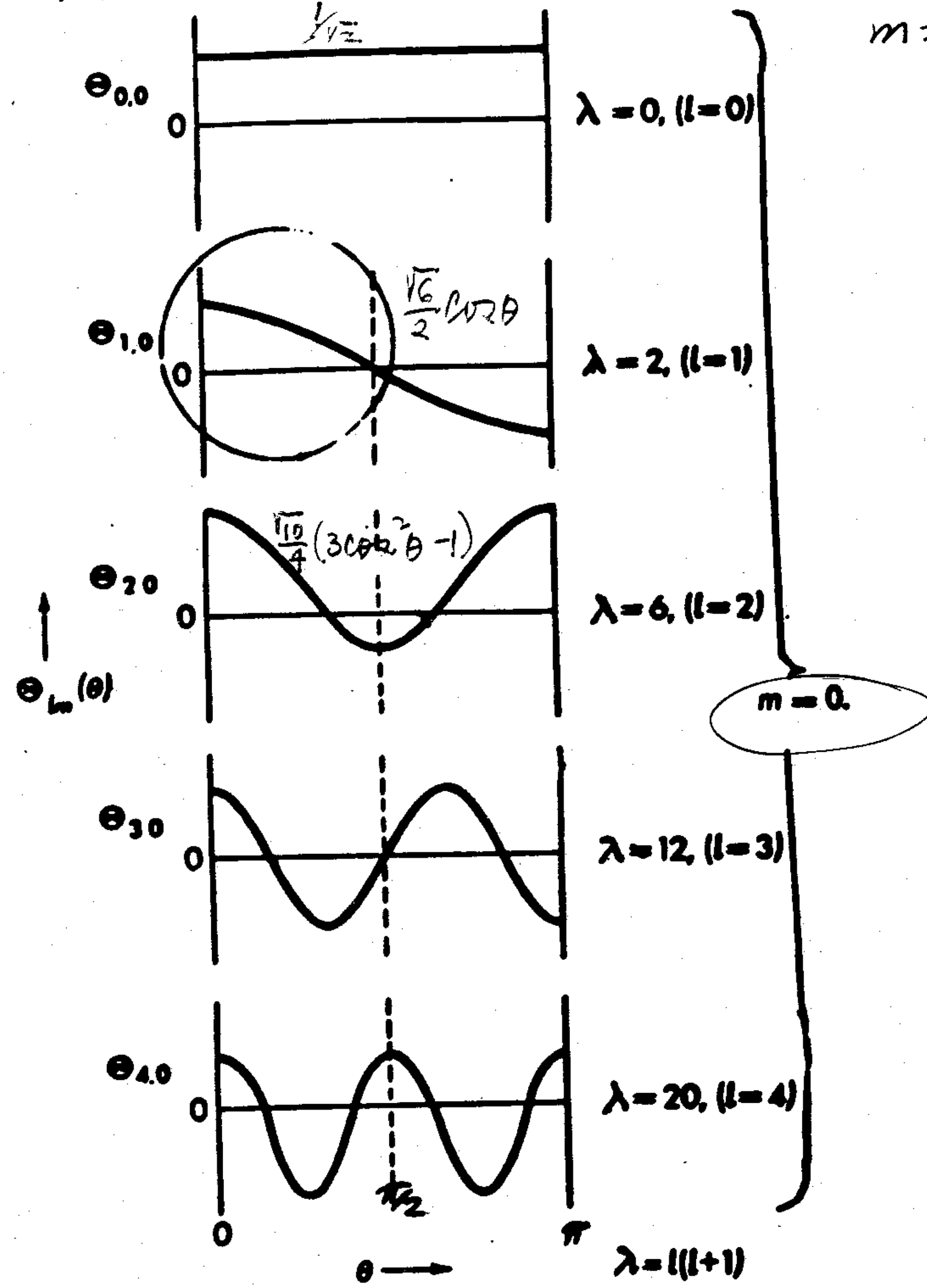
$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta)-\frac{m^2\,\Theta(\theta)}{\sin^2\theta} = -\frac{2Mr^2E}{\hbar^2}\Theta(\theta)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta = -\lambda\Theta$$

where $\lambda \equiv \frac{2Mr^2E}{\hbar^2}$

Consider solutions for $m=0$:

$m = 0, \pm 1, \pm 2 \ldots$

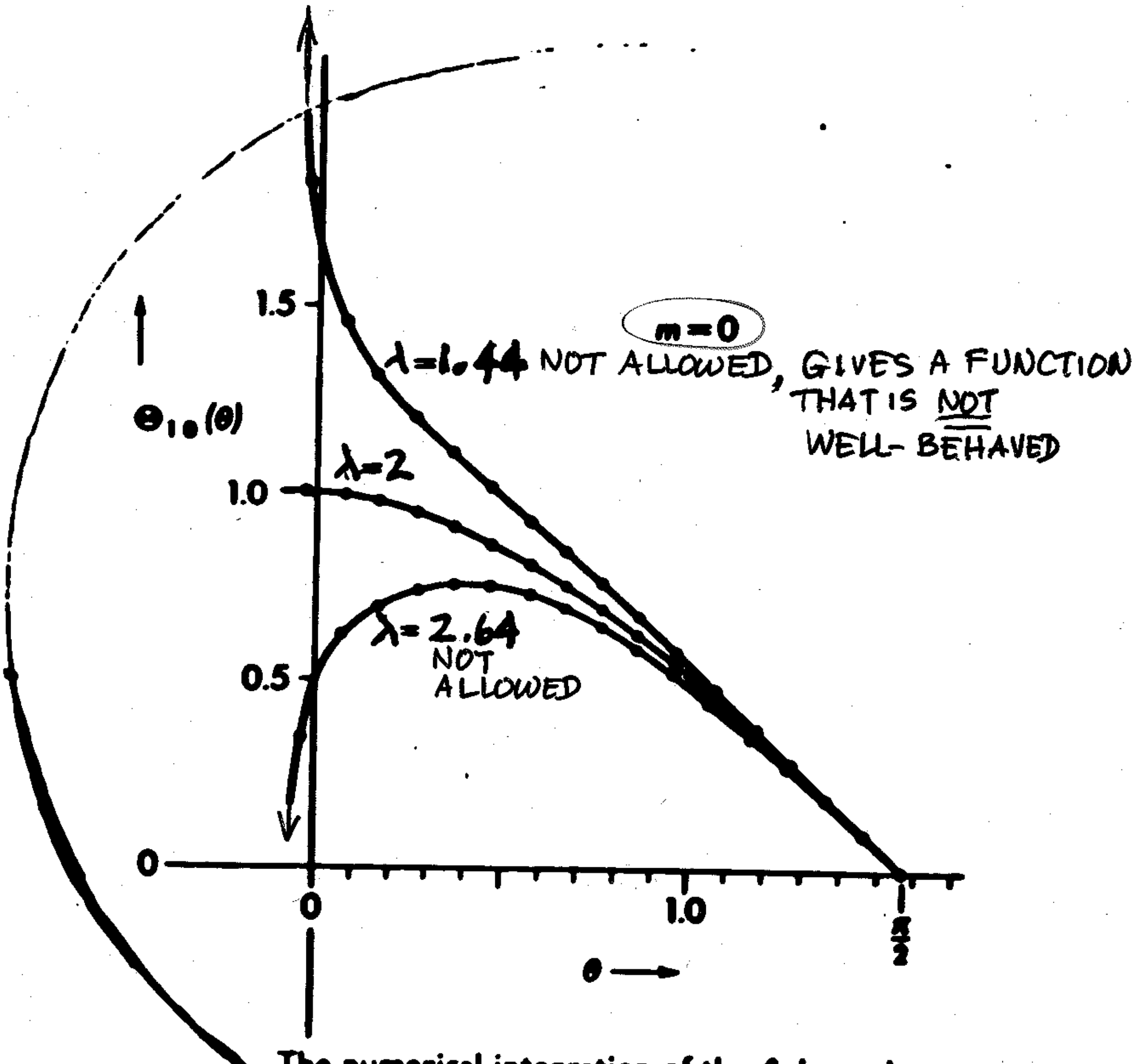


⋮

$\lambda = 30, 42,$ etc

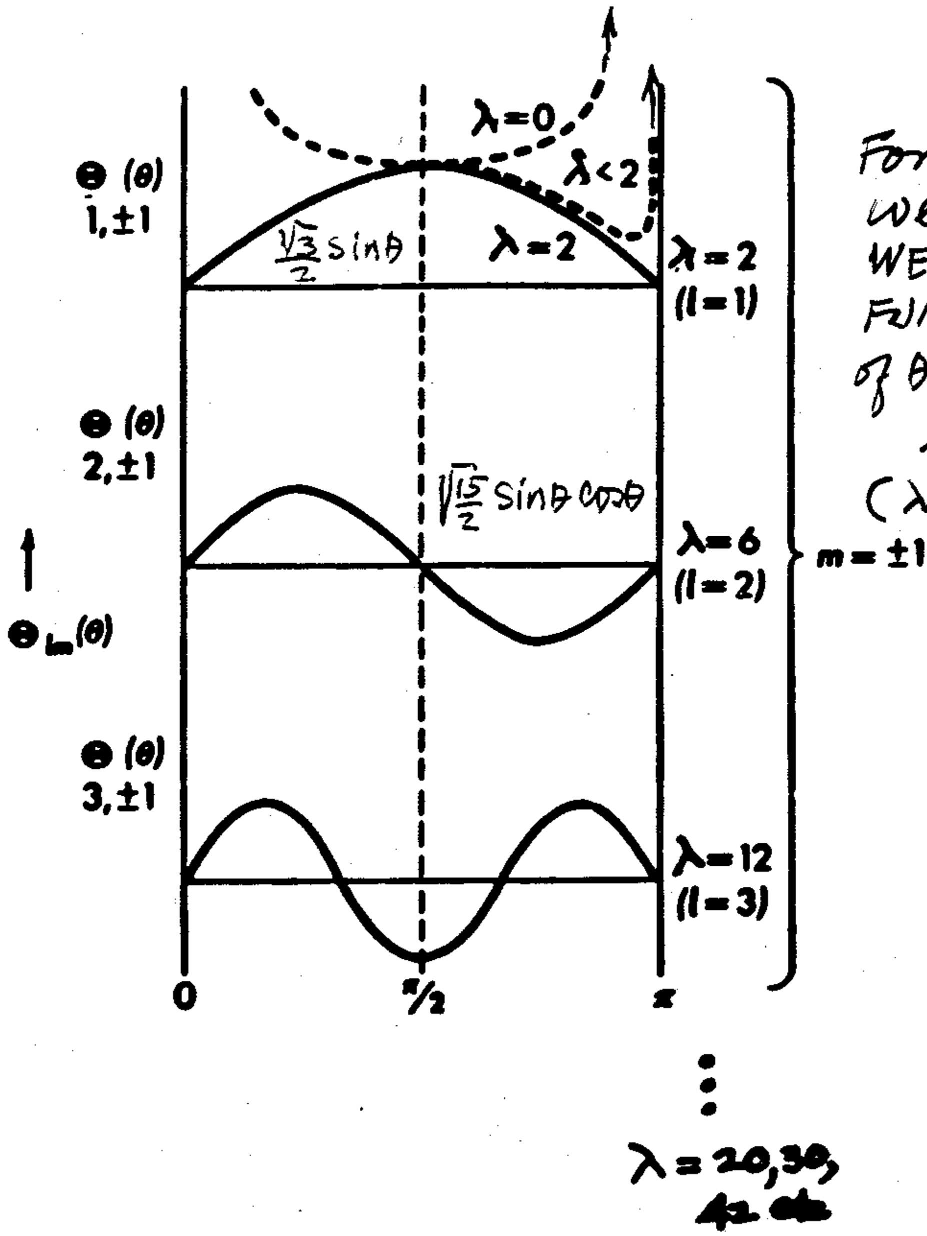
For $m=0$

We get a WELL-BEHAVED FUNCTION for $\lambda=2$

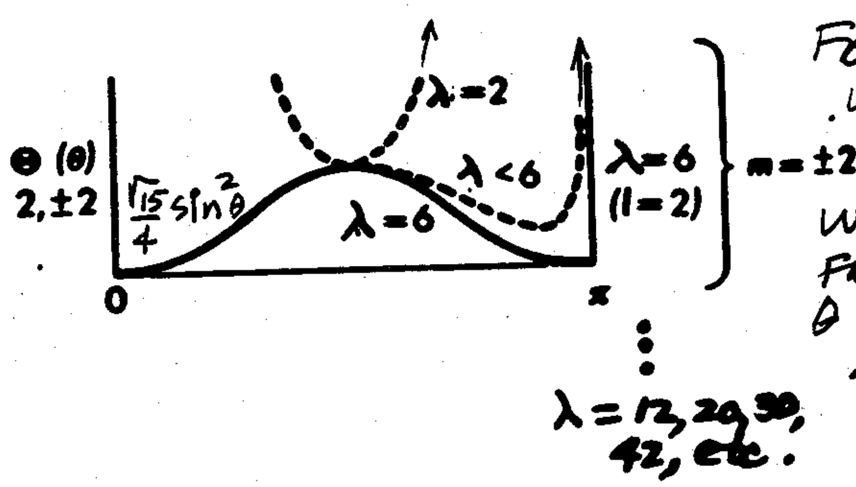


The numerical integration of the θ-dependent equation for $m=0$ and $\lambda=2$.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\lambda\Theta \qquad (\text{for } m=0)$$



For $m=\pm1$ we get WELL-BEHAVED FUNCTIONS of θ for $\lambda = 2, 6, 12, \ldots$ ($\lambda < 2$ NOT ALLOWED)



For $m=\pm2$ we get WELL-BEHAVED FUNCTIONS of θ for $\lambda = 6, \ldots$ ($\lambda < 6$ NOT ALLOWED)

Particle on a sphere, continued:

We have found that well-behaved functions of θ can be found only for

$m=0 \quad \lambda = 0, 2, 6, 12, 20, 30, 42, \ldots$

$m=\pm 1 \quad \lambda = 2, 6, 12, 20, 30, 42, \ldots$

$m=\pm 2 \quad \lambda = 6, 12, 20, 30, 42, \ldots$

$\vdots$

It looks like

$$\lambda = \ell(\ell+1)$$

where $\ell = 0, 1, 2, 3, \ldots$

Note also that

$\ell=0$ only goes with $m=0$ and that's all

$\ell=1$ only goes with $m=0, \pm 1$ only

$\ell=2$ only goes with $m=0, \pm 1, \pm 2$

$\vdots$

ℓ in general, goes with $m=0, \pm 1, \pm 2 \ldots \pm \ell$

The energy eigenvalues are:

$$E = \frac{\ell(\ell+1)\hbar^2}{2Mr^2} \qquad \text{from } \lambda \equiv \frac{2Mr^2 E}{\hbar^2} = \ell(\ell+1)$$

The hamiltonian eigenfunctions are:

$$Y_{\ell m}(\theta,\phi) = \Theta_{\ell m}(\theta) \cdot \underbrace{\Phi_m(\phi)}_{\frac{1}{\sqrt{2\pi}} e^{im\phi}}$$

for a particle constrained to move on the surface of a sphere.

$\Theta_{l|m|}(\theta)$ "ASSOCIATED LEGENDRE FUNCTIONS"

l	m	
0	0	$\Theta_{00} = \frac{1}{\sqrt{2}}$
1	0	$\Theta_{10} = \frac{\sqrt{6}}{2}\cos\theta$
1	±1	$\Theta_{11} = \frac{\sqrt{3}}{2}\sin\theta$
2	0	$\Theta_{20} = \frac{\sqrt{10}}{4}(3\cos^2\theta - 1)$
2	±1	$\Theta_{21} = \frac{\sqrt{15}}{2}\sin\theta\cos\theta$
2	±2	$\Theta_{22} = \frac{\sqrt{15}}{4}\sin^2\theta$

Particle on a circle: EIGENFUNCTIONS EIGENVALUES

$$H_{op} = \frac{-\hbar^2}{2Mr^2}\frac{d^2}{d\phi^2} \qquad \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi} \qquad E = \frac{m^2\hbar^2}{2Mr^2}$$

Z COMPONENT OF ANGULAR MOMENTUM $\quad l_{z\,op} = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \qquad m\hbar$

Particle on a sphere:

$$H_{op} = \frac{-\hbar^2}{2Mr^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\} = \frac{l^2_{op}}{2Mr^2} \qquad E = \frac{l(l+1)\hbar^2}{2Mr^2}$$

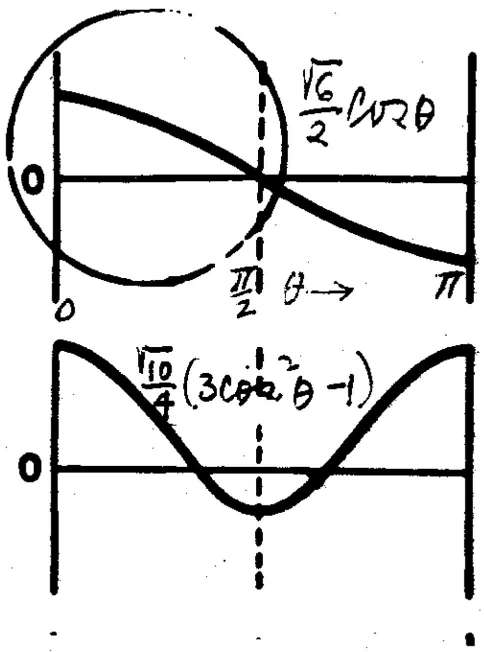
SQUARE OF THE ANGULAR MOMENTUM $\quad l^2_{op} = -\hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\} \qquad l(l+1)\hbar^2$

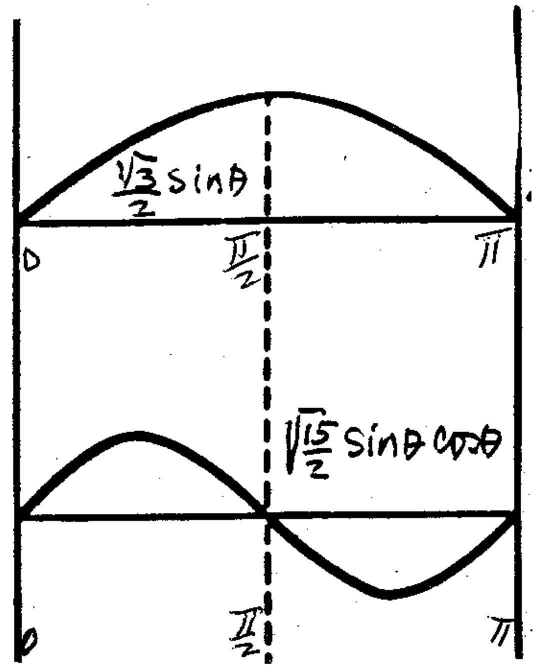
"SPHERICAL HARMONICS" $\quad Y_{lm}(\theta,\phi) = \Theta_{l|m|}(\theta)\cdot\Phi_m(\phi)$

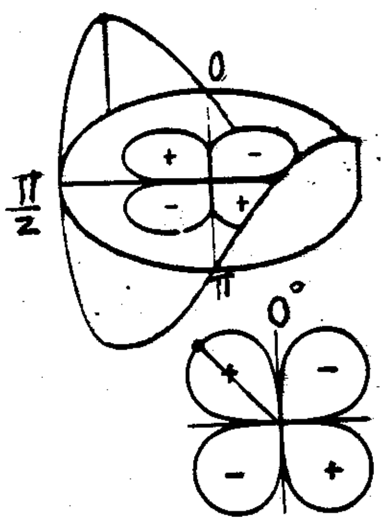
where $l = 0, 1, 2, 3, \ldots$
$m = 0, \pm1, \pm2, \ldots \pm l$

THUS, WE HAVE FOUND THE EIGENFUNCTIONS + EIGENVALUES

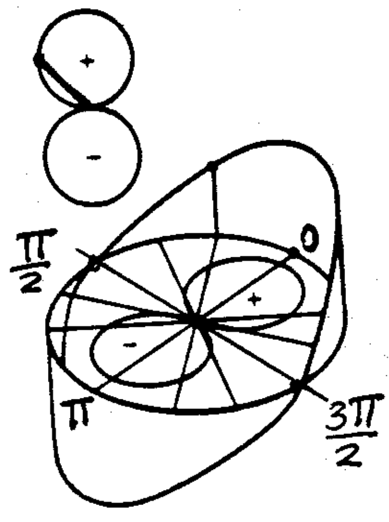
DIFFERENT WAYS OF PLOTTING functions of an angle







$\frac{1}{\sqrt{\pi}}\sin 2\phi$



$\frac{1}{\sqrt{\pi}}\cos\phi$

1. INTRODUCTION TO QUANTUM MECHANICS

2. ANGULAR MOMENTUM

2.1 Classical Mechanics → Quantum Mechanics

2.2 Commutation Rules of Angular Momentum

2.3 Example: Particle on a Sphere

2.4 Example: The Rigid Rotor

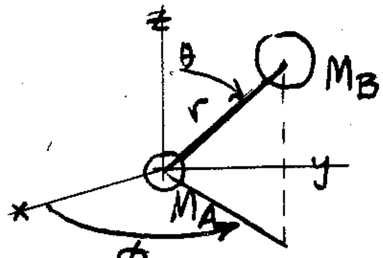
2.5 Eigenfunctions of Angular Momentum

EXAMPLE: The RIGID ROTOR: Two particles masses M_A and M_B at a fixed distance r from each other, zero potential energy

$$H_{total} = \frac{-\hbar^2}{2M_A}\nabla_A^2 - \frac{\hbar^2}{2M_B}\nabla_B^2$$

where $\nabla_A^2 \equiv \frac{\partial^2}{\partial x_A^2} + \frac{\partial^2}{\partial y_A^2} + \frac{\partial^2}{\partial z_A^2}$

SEPARATION of VARIABLES will be possible if we change into the center-of-mass and relative coordinates:



$$X_{CM} M_{total} = x_A M_A + x_B M_B$$
$$Y_{CM} M_{total} = y_A M_A + y_B M_B$$
$$Z_{CM} M_{total} = z_A M_A + z_B M_B$$

$x = x_B - x_A$
$y = y_B - y_A$
$z = z_B - z_A$

$M_{total} = M_A + M_B$

"REDUCED MASS" μ as in

$$\frac{1}{\mu} = \frac{1}{M_A} + \frac{1}{M_B}$$

Substitution into H_{total} leads to

$$H_{total} = \underbrace{\frac{-\hbar^2}{2M_{total}}\nabla_{CM}^2}_{\text{in coordinates of center of mass}} \quad \underbrace{-\frac{\hbar^2}{2\mu}\nabla^2}_{\text{in } x,y,z \text{ coordinates or } r,\theta,\phi}$$

SEPARABLE as follows:

$$\frac{-\hbar^2}{2M_{total}}\left(\frac{\partial^2}{\partial X_{CM}^2} + \frac{\partial^2}{\partial Y_{CM}^2} + \frac{\partial^2}{\partial Z_{CM}^2}\right) G(X_{CM}, Y_{CM}, Z_{CM}) = E_{transl}\, G(X_{CM}, Y_{CM}, Z_{CM})$$

$$\frac{-\hbar^2}{2\mu}\underbrace{\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}_{\nabla^2 \text{ for } r = \text{constant}} Y(\theta,\phi) = E_{rot}\, Y(\theta,\phi)$$

or $\frac{\ell^2}{2\mu r^2} Y(\theta,\phi) = E_{rot} Y(\theta,\phi)$

$$\boxed{E_{total} = E_{transl} + E_{rot}}$$

The translational motion is already solved. (This is a particle in a 3-dimensional box, except that the "particle" has mass $M_{total} = M_A + M_B$)

$$G(X_{CM}, Y_{CM}, Z_{CM}) = \sqrt{\frac{2}{L_1}} \sin\left(\frac{n_x \pi}{L_1} X_{CM}\right) \cdot \sqrt{\frac{2}{L_2}} \sin\left(\frac{n_y \pi}{L_2} Y_{CM}\right) \cdot \sqrt{\frac{2}{L_3}} \sin\left(\frac{n_z \pi}{L_3} Z_{CM}\right)$$

The rotational motion of the RIGID ROTOR has also already been solved. The hamiltonian looks exactly the same as that for a particle on a sphere, except that

PARTICLE on a SPHERE:

MOMENT OF INERTIA $= I = Mr^2$

$$H_{op} = \frac{\boldsymbol{\ell}^2}{2I} \qquad E = \frac{\ell(\ell+1)\hbar^2}{2I} \qquad Y_{\ell m}(\theta,\phi) = \Theta_{\ell |m|}(\theta) \cdot \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

RIGID ROTOR:

MOMENT OF INERTIA $= I = \mu r^2$

μ is "reduced mass," $\frac{1}{\mu} = \frac{1}{M_A} + \frac{1}{M_B}$

M	J	degeneracy	E
0, ±1, ±2, ±3, ±4	4	9	E = 20
0, ±1, ±2, ±3	3	7	E = 12
0, ±1, ±2	2	5	E = 6
M = 0, ±1	1	3	$2(\hbar^2/2I)$
M = 0	0	1	$E_{rot} = 0$

$$H_{op} = \frac{\boldsymbol{\ell}^2}{2I} \qquad E_{rot} = \frac{\ell(\ell+1)\hbar^2}{2I} \text{ except use more commonly } E_{rot} = \frac{J(J+1)\hbar^2}{2I}$$

and $Y_{JM}(\theta,\phi) = \Theta_{J|M|}(\theta) \cdot \frac{1}{\sqrt{2\pi}} e^{iM\phi}$

EIGENFUNCTIONS are the same "SPHERICAL HARMONICS" functions.

J is called the "rotational quantum number"

Note also that

PARTICLE on a CIRCLE:

MOMENT OF INERTIA $= I = Mr^2$

$$H_{op} = \frac{\boldsymbol{\ell}_z^2}{2I} \qquad E = \frac{m^2\hbar^2}{2I} \qquad \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

The DEGENERACY of each ROTATIONAL ENERGY LEVEL is $2J+1$, that is, there are $2J+1$ states of differing M values that have the same energy, just as the DEGENERACY of each energy level of the particle on a sphere is $2\ell+1$.

1. INTRODUCTION TO QUANTUM MECHANICS

2. ANGULAR MOMENTUM

2.1 Classical Mechanics → Quantum Mechanics

2.2 Commutation Rules of Angular Momentum

2.3 Example: Particle on a Sphere

2.4 Example: The Rigid Rotor

2.5 Eigenfunctions of Angular Momentum

1. INTRODUCTION TO QUANTUM MECHANICS

2. ANGULAR MOMENTUM

2.1 Classical Mechanics → Quantum Mechanics

2.2 Commutation Rules of Angular Momentum

2.3 Example: Particle on a Sphere

2.4 Example: The Rigid Rotor

2.5 Eigenfunctions of Angular Momentum

2.6 Raising and Lowering Operators

MORE on ANGULAR MOMENTUM:

Define the operators: (these are NOT Hermitian operators)

$\ell_+ \equiv \ell_x + i\ell_y$ **RAISING operator**

$\ell_- \equiv \ell_x - i\ell_y$ **LOWERING operator**

These operators commute with ℓ^2 but not with ℓ_z

$$[\ell_\pm, \ell^2] = 0$$

$$[\ell_\pm, \ell_z] = \mp\hbar\ell_\pm$$

Proof:

$$[\ell^2, \ell_+]\psi = \ell^2(\ell_x + i\ell_y)\psi - (\ell_x + i\ell_y)\ell^2\psi$$

$$= \underbrace{(\ell^2\ell_x - \ell_x\ell^2)}_{\text{zero}}\psi + i\underbrace{(\ell^2\ell_y - \ell_y\ell^2)}_{\text{zero}}\psi$$

$$[\ell^2, \ell_+] = 0$$

$$[\ell_+, \ell_z]\psi = (\ell_x + i\ell_y)\ell_z\psi - \ell_z(\ell_x + i\ell_y)\psi$$

$$= \underbrace{(\ell_x\ell_z - \ell_z\ell_x)}_{-i\hbar\ell_y}\psi + i\underbrace{(\ell_y\ell_z - \ell_z\ell_y)}_{i\hbar\ell_x}\psi$$

$$= -\hbar(\ell_x + i\ell_y)\psi$$

$[\ell_+, \ell_z] = -\hbar\ell_+$ Similarly, can show $[\ell_-, \ell_z] = +\hbar\ell_-$

Note that we can write l_x and l_y in terms of these RAISING and LOWERING operators:

$$l_+ \equiv l_x + i l_y$$

$$l_- \equiv l_x - i l_y$$

ADD: $$\frac{1}{2}(l_+ + l_-) = l_x$$

SUBTRACT: $$\frac{l_+ - l_-}{2i} = l_y$$

Example:
What is the result of applying l_+ to an eigenfunction of l_z?

$$l_+ Y_{lm}(\theta,\phi) = ?\ \text{function?}$$

Let us find out by applying the l_z operator on it:

$$l_z l_+ Y_{lm}(\theta,\phi) \underset{\substack{\text{commutator} \\ [l_+,\, l_z] = -\hbar l_+}}{\overset{\text{using}}{=}} (l_+ l_z + \hbar l_+) Y_{lm}(\theta,\phi)$$

$$= l_+ m\hbar Y_{lm}(\theta,\phi) + \hbar l_+ Y_{lm}(\theta,\phi)$$

$$l_z \boxed{l_+ Y_{lm}(\theta,\phi)} = (m+1)\hbar \boxed{l_+ Y_{lm}(\theta,\phi)}$$

THIS is an EIGENFUNCTION of l_z with eigenvalue $(m+1)\hbar$

Let us find out more by applying the l^2 operator on it:

$$l^2 \boldsymbol{l}_+ Y_{lm}(\theta,\phi) \underset{\text{COMMUTE}}{\overset{\text{THEY}}{=}} \boldsymbol{l}_+ l^2 Y_{lm}(\theta,\phi)$$

$$= \boldsymbol{l}_+ l(l+1)\hbar^2 Y_{lm}(\theta,\phi)$$

$$l^2 \boxed{\boldsymbol{l}_+ Y_{lm}(\theta,\phi)} = l(l+1)\hbar^2 \boxed{\boldsymbol{l}_+ Y_{lm}(\theta,\phi)}$$

THIS is an EIGENFUNCTION of l^2 with eigenvalue $l(l+1)\hbar^2$

Therefore,

$$\boldsymbol{l}_+ Y_{lm}(\theta,\phi) = \overbrace{N_{lm}}^{\text{a constant}} Y_{l\,m+1}(\theta,\phi)$$

an EIGENFUNCTION of $\boldsymbol{l}^2$ with eigenvalue $l(l+1)\hbar^2$
an EIGENFUNCTION of l_z with eigenvalue $(m+1)\hbar$

Example:
What is the result of applying $\boldsymbol{l}_-$ to an eigenfunction of l_z?

$$\boldsymbol{l}_- Y_{lm}(\theta,\phi) = \text{what function?}$$

Let us find out by applying the l_z operator:

$$l_z \boldsymbol{l}_- Y_{lm}(\theta,\phi) \underset{\text{commutator}}{\overset{\text{using}}{=}} \underbrace{(\boldsymbol{l}_- l_z - \hbar \boldsymbol{l}_-) Y_{lm}(\theta,\phi)}_{\boldsymbol{l}_- m\hbar Y_{lm} - \hbar \boldsymbol{l}_- Y_{lm}}$$

$$[l_-, l_z] = \hbar l_-$$

$$l_z \boxed{\mathbf{l}_- Y_{lm}(\theta,\phi)} = (m-1)\hbar \boxed{\mathbf{l}_- Y_{lm}(\theta,\phi)}$$

This is an EIGENFUNCTION of l_z with eigenvalue $(m-1)\hbar$

Also,

$$l^2 \mathbf{l}_- Y_{lm}(\theta,\phi) \underset{\text{commute}}{\overset{\text{they}}{=}} \mathbf{l}_- l^2 Y_{lm}(\theta,\phi) = \mathbf{l}_- l(l+1)\hbar^2 Y_{lm}(\theta,\phi)$$

$$l^2 \boxed{\mathbf{l}_- Y_{lm}(\theta,\phi)} = l(l+1)\hbar^2 \boxed{\mathbf{l}_- Y_{lm}(\theta,\phi)}$$

This is an EIGENFUNCTION of l^2 with eigenvalue $l(l+1)\hbar^2$

Therefore,

$$\mathbf{l}_- Y_{lm}(\theta,\phi) = \overbrace{N_{lm}^-}^{\text{a constant}} Y_{lm-1}(\theta,\phi)$$

an EIGENFUNCTION of l^2 with eigenvalue $l(l+1)\hbar^2$

an EIGENFUNCTION of l_z with eigenvalue $(m-1)\hbar$

Example:

What is the result of applying $\mathbf{l}_- \mathbf{l}_+$ to an eigenfunction of l_z?

$$\mathbf{l}_- \mathbf{l}_+ = (l_x - i l_y)(l_x + i l_y) =$$

$$= \underbrace{l_x^2 + l_y^2} + \underbrace{i l_x l_y - i l_y l_x}$$

$$= l^2 - l_z^2 + i\underbrace{[l_x, l_y]}_{i\hbar l_z}$$

$$\mathbf{l}_- \mathbf{l}_+ = l^2 - l_z^2 - \hbar l_z$$

Therefore, $\int Y_{lm}^* \mathbf{l}_- \mathbf{l}_+ Y_{lm}\, d\tau = l(l+1)\hbar^2 - m^2\hbar^2 - m\hbar^2$

$= [l(l+1) - m(m+1)]\hbar^2$

Now relate this to the constants N^+_{lm} and N^-_{lm}:

$$\int Y^*_{lm}\, \mathbf{l}_- \underbrace{\mathbf{l}_+ Y_{lm}}\, d\tau = \int Y^*_{lm}\, \mathbf{l}_- \left(N^+_{lm} Y_{lm+1}\right) d\tau$$

$$= N^+_{lm} \int Y^*_{lm}\, \underbrace{\mathbf{l}_- Y_{lm+1}}\, d\tau = N^+_{lm} \int Y^*_{lm} N^-_{lm+1} Y_{lm}\, d\tau$$

$$[l(l+1) - m(m+1)]\hbar^2 = N^+_{lm}\, N^-_{l\,m+1}$$

Question: What are these constants N^+_{lm} and N^-_{lm}?
We need one more relationship, which we can get as follows:

Consider the integral

$$\int Y^*_{lm}\, \underbrace{\mathbf{l}_- Y_{lm+1}}\, d\tau = N^-_{l\,m+1} \int Y^*_{lm} Y_{lm}\, d\tau = N^-_{l\,m+1}$$

$$\Downarrow \quad \mathbf{l}_- = l_x - i l_y$$

$$\int Y^*_{lm}\, l_x Y_{lm+1}\, d\tau - i \int Y^*_{lm}\, l_y Y_{lm+1}\, d\tau$$

$\downarrow$ Hermitian operator $\qquad\qquad$ $\downarrow$ Hermitian operator

$$\left(\int Y^*_{lm+1}\, l_x Y_{lm}\, d\tau\right)^* - i\left(\int Y^*_{lm+1}\, l_y Y_{lm}\, d\tau\right)^*$$

$$\left(\int Y^*_{lm+1}\, l_x Y_{lm}\, d\tau + i \int Y^*_{lm+1}\, l_y Y_{lm}\, d\tau\right)^*$$

$$\left(\int Y^*_{lm+1} (l_x + i l_y) Y_{lm}\, d\tau\right)^*$$

$$\left(\int Y^*_{lm+1}\, \underbrace{\mathbf{l}_+ Y_{lm}}\, d\tau\right)^*$$

$$\left(\int Y^*_{lm+1}\, N^+_{lm} Y_{lm+1}\, d\tau\right)^*$$

$$(N^+_{lm})^* \qquad = N^-_{l\,m+1}$$

Finally, we combine the two relationships that we have found:

$$\left.\begin{aligned}[\ell(\ell+1)-m(m+1)]\hbar^2 &= N^+_{\ell m}\ N^-_{\ell\, m+1}\\ (N^+_{\ell m})^* &= N^-_{\ell\, m+1}\end{aligned}\right\}$$

to get:

$$[\ell(\ell+1)-m(m+1)]\hbar^2 = N^+_{\ell m}\,(N^+_{\ell m})^*$$

$$\text{or}\quad N^+_{\ell m} = \sqrt{\ell(\ell+1)-m(m+1)}\;\hbar$$

and $[\ell(\ell+1)-m(m+1)]\hbar^2 = (N^-_{\ell m+1})^* N^-_{\ell m+1}$

from which we get, upon $m' = m+1$

$$[\ell(\ell+1)-(m'-1)m']\hbar^2 = (N^-_{\ell m'})^* N^-_{\ell m'}$$

$$\text{or}\qquad N^-_{\ell m} = \sqrt{\ell(\ell+1)-m(m-1)}\;\hbar$$

Summarizing:

$$N^{\pm}_{\ell m} = \sqrt{\ell(\ell+1)-m(m\pm 1)}\;\hbar$$

$$\boxed{\boldsymbol{\ell}_{\pm} Y_{\ell m} = \sqrt{\ell(\ell+1)-m(m\pm 1)}\;\hbar\, Y_{\ell\, m\pm 1}}$$

FIND ANY ONE EIGENFUNCTION of ℓ_z, FIND ALL OTHERS by $\boldsymbol{\ell}_{\pm}$

Note that if we apply the STEP DOWN op. to the function with the lowest eigenvalue of ℓ_z

$$\boldsymbol{\ell}_- Y_{\ell\, -\ell} \xrightarrow[\text{get}]{\text{we}} [\ell(\ell+1)-(-\ell)(-\ell-1)]^{1/2}\cdots = \text{ZERO}$$

Similarly, if we apply the STEP UP op. to the function with the highest eigenvalue of ℓ_z

$$\boldsymbol{\ell}_+ Y_{\ell\ell} \xrightarrow[\text{get}]{\text{we}} [\ell(\ell+1)-\ell(\ell+1)]^{1/2}\cdots = \text{ZERO}$$

IN GENERAL, for any ANGULAR MOMENTUM

whether or not it has a classical counterpart,

- We can denote the EIGENFUNCTIONS by specifying the quantum numbers associated with the SQUARE of the ANGULAR MOMENTUM and with the z-COMPONENT of the ANGULAR MOMENTUM

$$|j, m_j\rangle$$

- The EIGENVALUES are given by Postulate 2:

$$\mathbf{J}^2 |j, m_j\rangle = j(j+1)\hbar^2 |j, m_j\rangle$$

$$\mathbf{J_z} |j, m_j\rangle = m_j\hbar |j, m_j\rangle$$

- The LADDER OPERATORS are: (not Hermitian operators)

$$\mathbf{J_+} \equiv \mathbf{J_x} + i\mathbf{J_y} \qquad \mathbf{J_-} \equiv \mathbf{J_x} - i\mathbf{J_y}$$

such that

$$\underbrace{\langle j, m_j \pm 1|}_{\text{BRA}} \mathbf{J_\pm} \underbrace{|j, m_j\rangle}_{\text{KET}} = \left[j(j+1) - m_j(m_j \pm 1)\right]^{1/2} \hbar$$

THIS ASSURES THAT THE LADDER OF m_j VALUES RUNS IN UNIT STEPS from the TOP RUNG at $m_j = j$ DOWN to the BOTTOM RUNG at $m_j = -j$

- The COMMUTATION RULES are:

$$[\mathbf{J}^2, \mathbf{J_x}] = 0 \quad \text{and others}$$

$$[\mathbf{J_x}, \mathbf{J_y}] = i\hbar \mathbf{J_z} \quad \text{and others in cyclic order}$$

$$[\mathbf{J_\pm}, \mathbf{J}^2] = 0$$

$$[\mathbf{J_\pm}, \mathbf{J_z}] = \mp\hbar \mathbf{J_\pm}$$

- In other words, *if* the 3 components of an OBSERVABLE are represented by OPERATORS which satisfy the commutation relations:

$$[\mathbf{J}^2, \mathbf{J}_x] = 0 \qquad [\mathbf{J}^2, \mathbf{J}_y] = 0 \qquad [\mathbf{J}^2, \mathbf{J}_z] = 0$$

$$[\mathbf{J}_x, \mathbf{J}_y] = i\hbar \mathbf{J}_z$$ and others in cyclic order

then that OBSERVABLE is an ANGULAR MOMENTUM.

As a CONSEQUENCE of the above COMMUTATION RULES and the Hermitian property of the above operators, then IT FOLLOWS THAT:

$$\mathbf{J}^2 |j, m_j\rangle = j(j+1)\hbar^2 |j, m_j\rangle$$

and

$$\mathbf{J}_z |j, m_j\rangle = m_j\hbar\, |j, m_j\rangle$$

and

$$\langle j, m_j \pm 1 \,|\, \mathbf{J}_{\pm} \,|\, j, m_j\rangle = \left[j(j+1) - m_j(m_j \pm 1)\right]^{1/2} \hbar$$

in which

$$\mathbf{J}_{\pm} \equiv \mathbf{J}_x \pm i\mathbf{J}_y$$

which guarantees that the ladder of m_j values runs in UNIT STEPS from the top rung at $m_j = j$ down to the bottom rung at $m_j = -j$