About nuclear spins

Angular momentum in various systems that we have considered:

We have solved the Schrödinger equation for a ‘rigid rotor’ by recognizing that the same mathematical equation that we solved for the ‘single particle of mass m on the sphere’

$$(-\hbar^2/2mR^2)\ \{ (1/\sin\theta)(\partial/\partial\theta)\sin\theta\ \partial/\partial\theta + (1/\sin^2\theta)\partial^2/\partial\phi^2\}\ Y_{L,M}(\theta,\phi) = E_L\ Y_{L,M}(\theta,\phi)$$

and $E = (-\hbar^2/2mR^2)L(L+1)$

is the same mathematical equation we are solving for the internal motion of ‘two masses m_A and m_B connected by a massless rigid bar’, i.e., the ‘rigid rotor’:

$$(-\hbar^2/2\mu R^2)\ \{ (1/\sin\theta)(\partial/\partial\theta)\sin\theta\ \partial/\partial\theta + (1/\sin^2\theta)\partial^2/\partial\phi^2\}\ Y_{J,M}(\theta,\phi) = E_J\ Y_{J,M}(\theta,\phi)$$

where $(1/\mu) = (1/m_A) + (1/m_B)$ and $E = (-\hbar^2/2\mu R^2)J(J+1)$.

Also, we had found the operator $L^2 = -\hbar^2\{ (1/\sin\theta)(\partial/\partial\theta)\sin\theta\ \partial/\partial\theta + (1/\sin^2\theta)\partial^2/\partial\phi^2\}$ which permitted us to recognize that the Hamiltonian for the ‘single particle of mass m on the surface of a sphere’ could be written as $\mathcal{H}=L^2/2mR^2$

and the Hamiltonian for a ‘rigid rotor’ could be written as $\mathcal{H}=L^2/2\mu R^2$

The one-electron ‘(hydrogen-like) atom’ likewise presented us with again the same mathematical equation while solving

$$\{(-\hbar^2/2\mu)\ 1/r\ \partial^2/\partial r^2 + L^2/2\mu R^2 + V(r)\}\ F(r)\bullet Y_{\ell,m}(\theta,\phi) = E_n\ F(r)\bullet Y_{\ell,m}(\theta,\phi)$$

where $(1/\mu) = (1/m_e) + (1/m_p)$. In this atom the potential energy term is $V(r) = -Ze^2/r$.

Now let us consider a <u>model for the structure of a nucleus in terms of constituent protons and neutrons</u>. In this model, a nucleon (a proton or a neutron) is characterized by quantum numbers which arise in solving the wave equation for an individual nucleon bound in a nuclear potential well in the same way as for atomic electrons bound in the Coulomb field of a nucleus. If we assume this particular model, the Hamiltonian for a single nucleon is

$$\{(-\hbar^2/2m_n)\ 1/r\ \partial^2/\partial r^2 + L^2/2m_nR^2 + V(r)\}\ F(r)\bullet Y_{\ell,m}(\theta,\phi) = E_n\ F(r)\bullet Y_{\ell,m}(\theta,\phi)$$

where $V(r) = a + br^2+cr^3 + \ldots$, for example. The quantum numbers associated with one nucleon include ℓ, m_ℓ, s, m_s. A single proton or neutron has s = ½, so that m_s = +½, -½ . Thus, $m_j = m_\ell + ½$ and $m_j = m_\ell - ½$ and each nucleon could be considered in terms of the quantum numbers. ℓ, j, m_j . Keep the designations s, p, d, f, g,... for ℓ= 0, 1, 2, 3, ... Thus, the function for a single nucleon could be designated $d_{5/2}$ for ℓ= 2, $m_j = m_\ell + ½ = 2 +½ = 5/2$ which would be associated with the function $F(r)\,|\,\ell, j, m_j\rangle = F(r)\,|\,2, 5/2, 5/2\rangle$ or $d_{5/2}$. The ordering of the energies of a nucleon will, as in many-electron atoms, depend on the form of V(r) used. One example is shown in the table on the next page for V(r) = rectangular well. Where all components of the angular momentum are included we have what might be considered “closed shells”.

	ℓ	(max m_ℓ) ± ½	max m_j	j	designation	no. of states	total	Z
1g	ℓ= 4	4 +½ =	9/2	j = 9/2	$1g_{9/2}$	10	22	50
2p	ℓ= 1	1 - ½ =	1/2	j =1/2	$2p_{1/2}$	2		40
2p	ℓ= 1	1 + ½ =	3/2	j =3/2	$2p_{3/2}$	4		38
1f	ℓ= 3	3 - ½ =	5/2	j =5/2	$1f_{5/2}$	6		34
1f	ℓ= 3	3+ ½ =	7/2	j =7/2	$1f_{7/2}$	8	8	28
2s	ℓ= 0	0 + ½ =	½	j =½	$2s_{1/2}$	2	6	20
1d	ℓ= 2	2 - ½ =	3/2	j =3/2	$1d_{3/2}$	4		18
1d	ℓ= 2	2 + ½ =	5/2	j =5/2	$1d_{5/2}$	6	6	14
1p	ℓ= 1	1 - ½ =	½	j =½	$1p_{1/2}$	2		8
1p	ℓ= 1	1 + ½ =	3/2	j =3/2	$1p_{3/2}$	4	6	6
1s	ℓ= 0	0 + ½ =	½	j =1/2	$1s_{1/2}$	2	2	2

Configurations of protons and neutrons that may be used to determine the overall angular momentum (spin I) of a nucleus:

no. of protons	proton config	neutron config	nucleus	spin I
Z = 1	$(1s_{1/2})$	$(1s_{1/2})^2$	^{3}H	½
Z = 1	$(1s_{1/2})$	$(1s_{1/2})$	^{2}H	1
Z = 2	$(1s_{1/2})^2$	$(1s_{1/2})^2$	^{4}He	0
Z = 2	$(1s_{1/2})^2$	$(1s_{1/2})$	^{3}He	½
Z = 6	$(1s_{1/2})^2(1p_{3/2})^4$	$(1s_{1/2})^2(1p_{3/2})^4$	^{12}C	0
Z = 6	$(1s_{1/2})^2(1p_{3/2})^4$	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})$	^{13}C	½
Z = 8	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2$	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2$	^{16}O	0
Z = 8	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2$	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})$	^{17}O	5/2
Z = 14	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^6$	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^6$	^{28}Si	0
Z = 14	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^6$	14 + $(2s_{1/2})$	^{29}Si	½
Z = 15	14+$(2s_{1/2})$	14 + $(2s_{1/2})^2$	^{31}P	½
Z = 16	14+$(2s_{1/2})^2$	14 + $(2s_{1/2})^2$	^{32}S	0
Z = 16	14+$(2s_{1/2})^2$	14+ $(2s_{1/2})^2(1d_{3/2})$	^{33}S	3/2

When two or more nucleons aggregate to form a nucleus, the quantum state of the system as a whole is called a nuclear level. Each nuclear level is characterized by a particular value of the total angular momentum that arises from the adding of angular momentum vectors of the nucleons. The coupling of the angular momenta is not simple, because the actual individual motions of the nucleons must be strongly interdependent because of the small distances. Nevertheless, one good model is to assume the same kind of coupling scheme as for electrons in an atom. The addition of angular momenta is easy to predict when there is only one nucleon outside of a "closed shell", as in some examples shown in the above table.